Solutions for Recurrence Relations using Master Theorem

CSE2003 Data Structures and Algorithms

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Masters theorem

→ This method is mainly applied to the reccurence relation's of the form;

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ & $b > 1$ are constants and $f(n)$ is an asymptotically positive function.

This theorem has 3 cases;

i) if $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

ii) if $f(n) = \Theta(n^{\log_b a} \log^k n)$ with $k \geq 0$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

iii) if $f(n) = \Omega(n^{\log_b a + \epsilon})$ with $\epsilon > 0$ and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$.

Regularity condition: $a f(n/b) \leq c f(n)$ for some constant $c < 1$ & all sufficiently large n.

Simplified for cases;

$$T(n) = aT(n/b) + \Theta(n^k \log^p n)$$

$a \geq 1$, $b > 1$, $k \geq 0$ & p is a real number.

i) if $a > b^k$, then $T(n) = \Theta(n^{\log_b a})$

ii) if $a = b^k$,

a) if $p > -1$, then $T(n) = \Theta(n^{\log_b a} \log^{p+1} n)$

b) if $p = -1$, $T(n) = \Theta(n^{\log_b a} \log \log n)$

c) if $p < -1$, $T(n) = \Theta(n^{\log_b a})$

iii) if $a < b^k$

a) if $p \geq 0$, then $T(n) = \Theta(n^k \log^p n)$

b) if $p < 0$, then $T(n) = O(n^k)$

And regularity condition satisfaction.

Problems

Q. $T(n) = 4T(n/2) + n^2$

here this reccurence relation is of the form $T(n) = aT(n/b) + f(n)$

So master's theorem can be applied

$\therefore$ checking the suitable cases for final result.

here $a = 4$; $b = 2$, $a \geq 1$ & $b > 1$

and $f(n) = n^2$

$f(n) = \Theta(n^{\log_b a} \cdot \log^k n)$, where $k \geq 0$

$\therefore a = 4, b = 2, k = 0$

$f(n) = n^2$

So this satisfies case 2

$\therefore T(n) = \Theta(n^{\log_b a} \cdot \log^{k+1} n)$
from case 2

$T(n) = \Theta(n^{\log_2 4} \cdot \log^{0+1} n)$

$\therefore T(n) = \Theta(n^2 \cdot \log n)$

Q. $T(n) = 4T(n/2) + n^2 \log n$

It's of the form, $T(n) = aT(n/b) + f(n)$

So masters theorem can be applied

here $a = 4$, $b = 2$, $f(n) = n^2 \log n$

Checking for the cases,

Case 2:

$f(n) = \Theta(n^{\log_b a} \cdot \log^k n)$, $k \geq 0$

So substituting a, b, k = 1

$f(n) = \Theta(n^{\log_2 4} \cdot \log^1 n)$

$= n^2 \cdot \log n$

$\therefore$ It satisfies case 2 of masters theorem

$\therefore$ $T(n) = \Theta(n^{\log_b a} \cdot \log^{k+1} n)$

here $a = 4$, $b = 2$, $k = 1$

$\therefore$ $T(n) = \Theta(n^{\log_2 4} \cdot \log^{1+1} n)$

$T(n) = \Theta(n^2 \cdot \log^2 n)$

Q. $T(n) = 8T(n/2) + \Theta(n^2)$

It is of the form, $T(n) = aT(n/b) + f(n)$

$\therefore$ masters theorem can be applied.

here $a = 8$, $b = 2$, $f(n) = \Theta(n^2)$

So for checking suitable case.

case 1, $f(n) = O(n^{\log_b a - \varepsilon})$, $\varepsilon > 0$, then

$T(n) = \Theta(n^{\log_b a})$

$\therefore O(n^{\log_b a - \varepsilon})$ $a = 8$, $b = 2$

$\Rightarrow O(n^{\log_2 8 - 1})$, $\varepsilon = 1$

$\Rightarrow O(n^{3-1}) = O(n^2)$

$\therefore$ It satisfies the case 1

$\Rightarrow T(n) = \Theta(n^{\log_b a})$

$\Rightarrow \Theta(n^{\log_2 8})$ $\left[\begin{array}{l} a = 8 \\ b = 2 \end{array}\right)$

$T(n) = \Theta(n^3)$

Q. $T(n) = 16T(n/4) + n$

Sol:- Here its of the form $aT(n/b) + f(n)$, so masters theorem is suitable.

here $a = 16$, $b = 4$, $f(n) = n$

so checking for case 1, as $f(n) = n$

$\therefore f(n) = O(n^{\log_b a - \varepsilon})$, $\varepsilon > 0$,

$O(n^{\log_4 16 - \varepsilon})$ when, $\varepsilon = 1$

$O(n^{\log_4 16 - 1}) \rightarrow O(n)$

$\Rightarrow$ It satifies case 1.

$\therefore T(n) = \Theta(n^{\log_b a})$

$T(n) = \Theta(n^{\log_4 16})$ [as $a = 16$, $b = 4$

$T(n) = \Theta(n^2)$

Q. $T(n) = 4T(n/2) + \log n$

Sol:- This relation is also of the form $aT(n/b) + f(n)$, so masters theorem is applicable.

here $a = 4$, $b = 2$, $f(n) = \log n$

We can apply simplified cases,

$\therefore$ $a = 4$, $b = 2$, $k = 0$, $p = 1$

$\therefore$ $a > b^k$

So it satisfies case 1 of masters theorem

$T(n) = \Theta(n^{\log_b a})$

$T(n) = \Theta(n^{\log_2 4})$

$T(n) = \Theta(n^2)$

Q. $T(n) = 2T(n/2) + n\log n$

Sol:- It is of the form $aT(n/b) + f(n)$

So applying masters theorem

here $a = 2$, $b = 2$, $f(n) = n\log n$

So checking the cases suitable,

In case 2, $f(n) = \Theta(n^{\log_b a} \log^k n)$ where k is some constant, $k \geq 0$

$f(n) = \Theta(n^{\log_2 2} \cdot \log^1 n)$ $[a = b = 2, k = 1]$

$f(n) = \Theta(n^1 \cdot \log n)$

$\therefore$ It satisfies case 2 then

$T(n) = \Theta(n^{\log_b a} \cdot \log^{k+1} n)$

$T(n) = \Theta(n^{\log_2 2} \cdot \log^{1+1} n)$

$T(n) = \Theta(n \cdot \log^2 n)$

Q. $T(n) = 3T(n/2) + n$

It is also in the form of $aT(n/b) + f(n)$

here $a = 3$, $b = 2$, $f(n) = n$

We can apply simplified cases of masters theorem

$\therefore$ $a = 3$, $b = 2$, $k = 1$, $p = 0$ (real number)

$\Rightarrow a > b^k$

$\therefore$ It satisfies case 1 of masters theorem

$\Rightarrow T(n) = \Theta(n^{\log_b a})$

$T(n) = \Theta(n^{\log_2 3})$

Q. $T(n) = T(n/2) + n(2 - \cos n)$

Sol:- By using Masters theorem.

We can use simplified cases for choosing the case it suites.

here $a = 1$, $b = 2$, $k = 1$

$\therefore a < b^k$

$\therefore$ it comes under case 3 of master theorem

$\therefore$ For satisfying it, it should satisfy regularity condition.

As in $f(n)$, cosine function comes into action,

$\therefore$ when $n = 2\pi k$, where k is odd and arbitarily large.

So for any value of n, the value of $c \geq 3/2$

$\Rightarrow$ it violates the regularity condition

$\therefore$ Masters theorem cannot be applied.

As it fails in case 3

Q. $T(n) = 3T(n/4) + n\log n$.

Sol:- It is also of the form $aT(n/b) + f(n)$

so masters theorem can be applied.

here $a=3$, $b=4$, $f(n) = n\log n$

So checking the cases,

In case 3, $f(n) = \Omega(n^{\log_b a + \varepsilon})$ with $\varepsilon > 0$,

And also by using simplified cases

$a=3$, $b=4$, $k=1$

here $a < b^k$

$\therefore$ It comes under case 3

To satisfy, checking for regularity condition

$a.f(n/b) \leq c.f(n)$ where $c<1$ & c is some constant.

For regularity, $af(n/b) = 3 \times (n/4)\log(n/4)$

$\therefore$ let $k = n/4$ and observe that $af(k)$ is equal to $ak\log k$

$\Rightarrow 3(n/4)\log(n/4) \leq (3/4)n\log n = cf(n)$

$\therefore c = 3/4$

$\therefore$ It satisfies the case 3 then

$T(n) = \Theta(n\log n)$

Q. $T(n) = 3T(n/3) + n/2$

Sol:- It is of the form $aT(n/b) + f(n)$, so Master's theorem can be applied;

here $a = 3$, $b = 3$, $f(n) = n/2$

Checking the cases,

In case 2, $f(n) = \Theta(n^{\log_b a} \cdot \log^k n)$,

where k is some constant, $k \geq 0$

when $a = b = 3$, $f(n) = \Theta(n)$

$\therefore$ It satisfies case 2

When verify using simplified cases

$a = 3$, $b = 3$, $k = 1$

$a = b^k \rightarrow$ it satisfies case 2

$\therefore T(n) = \Theta(n^{\log_b a} \cdot \log^{k+1} n)$

$T(n) = \Theta(n^{\log_3 3} \cdot \log^{0+1} n)$

$T(n) = \Theta(n \log n)$

Q. $T(n) = 0.5\, T(n/2) + 1/n$

Sol:- In this reccurence relation, when compared to the masters theorem condition;

Relation of the form $aT(n/b) + f(n)$

& $a \geq 1$, $b > 1$, $f(n)$ is asymptotically positive function

In first condition $a \geq 1$, it violates the condition ie $a < 1$

In this the master theorem is violated and cannot be applied.

Q. $T(n) = 2T(n/2) + n^2$

Sol:- It is of the form $aT(n/b) + f(n)$, So we can use master's theorem

here $a = 2$, $b = 2$, $f(n) = n^2$

$\log_b a = \log_2 2 = 1$

So checking the cases

In case 3, $f(n) = \Omega(n^{\log_b a + \varepsilon})$ with $\varepsilon > 0$ and $f(n)$ satisfies regularity condition then $T(n) = \Theta(f(n))$

when $\varepsilon = 1$

$$f(n) = \Omega(n^{\log_2 2 + 1}) = \Omega(n^2)$$

∴ It comes under case 3, So regularity condition. $a f(n/b) \leq c \cdot f(n)$

$$\therefore 2 \times (n/2)^2 \leq c \cdot n^2 \quad \forall n \geq 1$$

∴ It satisfies the case 3 then,

$$T(n) = \Theta(n^{\log_b a})$$

$$T(n) = \Theta(n^2)$$

Q. $T(n) = 2T(n/2) + \theta(n)$

Sol:- It is of the form $aT(n/b) + f(n)$

here $a = 2$, $b = 2$, $f(n) = \theta(n)$

So applying the cases,

In case 2, $\theta(n)$ ie $f(n) = \theta(n^{\log_b^a} \cdot \log^k n)$

where k is constant, $k \geq 0$

$\therefore f(n) = \theta(n^{\log_b^a} \cdot \log^k n)$ $\left[\begin{array}{l} a = b = 2 \\ k = 0 \end{array} \right.$

$= \theta(n^{\log_2^2} \cdot \log^0 n)$

$\Rightarrow \theta(n)$

$\therefore$ It satisfies case 2 of Masters theorem.

$\Rightarrow T(n) = \theta(n^{\log_b^a} \cdot \log^{k+1} n)$

$T(n) = \theta(n^{\log_2^2} \cdot \log^{0+1} n)$

$T(n) \Rightarrow \underline{\underline{\theta(n \cdot \log n)}}$

Q. $T(n) = 4T(n/3) + n$

Sol:- This is in the form $aT(n/b) + f(n)$, so masters theorem is applicable.

here $a = 4$, $b = 3$, $f(n) = n$

We can solve this using simplified cases,

$\therefore a = 4$, $b = 3$, $k = 1$, $p = 0$

$\Rightarrow a > b^k$

$\therefore$ It satisfies case 1 of masters theorem

$\Rightarrow T(n) = \Theta(n^{\log_b a})$

here $a = 4$, $b = 3$

$\therefore T(n) = \Theta(n^{\log_3 4})$

$T(n) = \Theta(n^{1.261})$

Q. $T(n) = 3T(n/3) + \sqrt{n}$

Sol:- It is also in the form $aT(n/b) + f(n)$

here $a = b = 3$

So checking the cases;

$f(n) = n^{1/2}$

In case 2, $f(n) = \Theta(n^{\log_b a} \cdot \log^k n)$

It doesnt satisfy

$\therefore$ In case 1, $f(n) = O(n^{\log_b a - \varepsilon})$

where $\varepsilon > 0$

Taking $\varepsilon = 1/2$

$\therefore f(n) = O(n^{\log_3 3 - 1/2})$

$= O(n^{1 - 1/2}) = O(n^{1/2})$

$\Rightarrow$ It satisfies case 1 of masters theorem.

$\therefore T(n) = \Theta(n^{\log_b a})$

$T(n) = \Theta(n^{\log_3 3})$

$T(n) = \Theta(n)$

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