Solutions for Recurrence Relations using Substitution Method

CSE2003 Data Structures and Algorithms

Instructor : Dr. C. Oswald

Students : S. Girish and Harshanth K Prakash

Vellore Institute of Technology Chennai

(An Institution of Eminence recognized by MHRD)

chennai.vit.ac.in

1.) $T(n) = \begin{cases} T(\frac{n}{2}) + cn^2 &, n \geq 2 \\ c &, n = 1 \end{cases}$ — ①

A: Backward Substitution:

Let's replace n with n/2 in the previous equation.

$$T(\tfrac{n}{2}) = T(\tfrac{n}{4}) + c(\tfrac{n}{2})^2 \quad — ②$$

where, c is a constant.

Now, substituting ② in ①, we get:

$$T(n) = T(\tfrac{n}{2}) + cn^2$$

$$= \underbrace{\left[T(\tfrac{n}{4}) + c(\tfrac{n}{2})^2\right]}_{\text{value of } T(\frac{n}{2})} + cn^2 \quad — ③$$

Again substituting $T(\frac{n}{4})$ value in place of n in ①:

$$T(\tfrac{n}{4}) = T(\tfrac{n}{8}) + c(\tfrac{n}{4})^2$$

Putting this value in ③, we get:

$$T(n) = cn^2 + c(\tfrac{n}{2})^2 + c(\tfrac{n}{4})^2 + T(\tfrac{n}{8}) \quad — ④$$

Since, we are breaking our problem int 2 subproblems of half size, we can assume that n is in the form of 2^k.

It might be possible that n is not exactly in the form of 2^k but $2^k \pm k_0$, where k_0 is some constant.

But we are concerned about the rate of the growth only and this approximation is going to give us that.

So, we will proceed by assuming that n is in the form of 2^k.

Recalling the eqn. ④ :

$$T(n) = cn^2 + c\left(\frac{n}{2}\right)^2 + c\left(\frac{n}{4}\right)^2 + T\left(\frac{n}{8}\right)$$

$$= cn^2\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2\right] + T\left(\frac{n}{8}\right)$$

$$= cn^2\left[1 + \left(\frac{1}{2^1}\right)^2 + \left(\frac{1}{2^2}\right)^2\right] + T\left(\frac{n}{2^3}\right)$$

$$= cn^2\left[\frac{1}{4^0} + \frac{1}{4^1} + \frac{1}{4^2}\right] + T\left(\frac{n}{2^3}\right)$$

$$\Rightarrow T(n) = cn^2\left[\frac{1}{4^0} + \frac{1}{4^1} + \frac{1}{4^2} + \frac{1}{4^3}\right] + T\left(\frac{n}{2^4}\right)$$

Similarly, we can write :

$$T(n) = cn^2\left[\frac{1}{4^0} + \frac{1}{4^1} + \frac{1}{4^2} + \cdots \frac{1}{4^{k-1}}\right] + T\left(\frac{n}{2^k}\right)$$

Since $n = 2^k$,

$$T(n) = cn^2\underbrace{\left[\frac{1}{4^0} + \frac{1}{4^1} + \frac{1}{4^2} + \cdots \frac{1}{4^{k-1}}\right]}_{k \text{ times}} + T\left(\frac{2^k}{2^k}\right)$$

[So, using the formula for sum of terms in a G.P. :

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \frac{(a)(1 - r^n)}{(1 - r)} \quad (\text{if } r < 1)$$

Here, $a = 1$; $r = \frac{1}{4}$; $n = k$]

$$\Rightarrow T(n) = cn^2\left[\frac{(1)\left(1 - \left(\frac{1}{4}\right)^k\right)}{\left(1 - \frac{1}{4}\right)}\right] = cn^2\left[\frac{1 - \left(\frac{1}{4}\right)^k}{\frac{3}{4}}\right]$$

$$T(n) = \frac{4}{3}cn^2\left[1 - \left(\frac{1}{4}\right)^k\right] + T(1) \quad \text{— ⑤}$$

As mentioned above, T(1) is the best case.
So, we have reached the base case.
Also,

$$\boxed{n = 2^k}$$

$$\Rightarrow \log_2(n) = \log_2(2^k)$$

$$\Rightarrow \boxed{\lg(n) = k}$$

Replacing the value of k in ⑤, we get :

$$T(n) = \frac{4}{3}cn^2\left[1-\left(\frac{1}{4}\right)^{\lg_2 n}\right] + T(1)$$

$$= \frac{4}{3}cn^2\left[1-(2^{-2})^{\lg_2 n}\right] + T(1) \left[\because \frac{1}{4} = \frac{1}{2^2} = 2^{-2}\right]$$

$$= \frac{4}{3}cn^2\left[1-2^{-2\cdot\log_2 n}\right] + T(1) \left[\because (a^m)^n = a^{mn}\right]$$

$$= \frac{4}{3}cn^2\left[1-2^{\log_2 n^{-2}}\right] + T(1)$$

$$= \frac{4}{3}cn^2\left[1-n^{-2}\right] + T(1) \left[\because a^{\log_a n} = n\right]$$

$$= \frac{4}{3}cn^2\left[1-\frac{1}{n^2}\right] + T(1)$$

$$= \frac{4}{3}cn^2 - \frac{4}{3}\cancel{cn^2}\left(\frac{1}{\cancel{n^2}}\right) + T(1)$$

$$= \frac{4}{3}cn^2 - \frac{4}{3}c + T(1) = \frac{4}{3}cn^2 - \frac{4}{3}c + c$$

$$\boxed{T(n) = \Theta(n^2).}$$

2.) $T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + cn^2 & , n \geq 2 \\ c & , n = 1 \end{cases}$

A: $T(n) = 2T\left(\frac{n}{2}\right) + cn^2$

$= 2\left[2T\left(\frac{n}{4}\right) + c\left(\frac{n}{2}\right)^2\right] + cn^2$

$= 4T\left(\frac{n}{4}\right) + c(2)\left(\frac{n}{2}\right)^2 + cn^2$

$= 4\left[2T\left(\frac{n}{8}\right) + c\left(\frac{n}{4}\right)^2\right] + c(2)\left(\frac{n}{2}\right)^2 + cn^2$

$= 8T\left(\frac{n}{8}\right) + c(4)\left(\frac{n}{4}\right)^2 + c(2)\left(\frac{n}{2}\right)^2 + cn^2$

$= 2^3T\left(\frac{n}{2^3}\right) + c\left[\left(4 \times \frac{n}{4} \times \frac{n}{4}\right) + \left(2 \times \frac{n}{2} \times \frac{n}{2}\right) + n^2\right]$

$= 2^3T\left(\frac{n}{2^3}\right) + c\left[\frac{n^2}{4} + \frac{n^2}{2} + n^2\right]$

$= 2^3T\left(\frac{n}{2^3}\right) + cn^2\left[\frac{1}{4} + \frac{1}{2} + 1\right]$

$= 2^3T\left(\frac{n}{2^3}\right) + cn^2\left[1 + \frac{1}{2} + \frac{1}{4}\right]$

$= 2^4T\left(\frac{n}{2^4}\right) + cn^2\left[1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right]$

Let's assume that n is of the form, 2^k.

$\Rightarrow T(n) = cn^2\left[\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \frac{1}{2^{k-1}}\right] + 2^kT\left(\frac{n}{2^k}\right)$

Since, $\boxed{n = 2^k}$

$T(n) = cn^2\left[\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \frac{1}{2^{k-1}}\right] + \left[T\left(\frac{2^k}{2^k}\right)\right] * 2^k$

[Sum of terms in a G.P. $= \frac{(a)(r^n - 1)}{(r-1)}$ $(r > 1)$

$= \frac{(a)(1 - r^n)}{(1-r)}$ $(r < 1)$

Here, $a = 1$; $r = \frac{1}{2}$; $n = k$]

$$\Rightarrow T(n) = cn^2\left[\frac{(1)\left(1-\frac{1}{2^k}\right)}{\left(1-\frac{1}{2}\right)}\right] + 2^k T(1)$$

$$= cn^2\left[\frac{\left(1-\frac{1}{2^k}\right)}{\left(\frac{1}{2}\right)}\right] + 2^k T(1)$$

$$= 2cn^2\left(1-\frac{1}{2^k}\right) + 2^k.c \left[\because T(1)=c\right]$$

Also,

$$n = 2^k$$

$$\Rightarrow \log_2(n) = \log_2(2^k)$$

$$\Rightarrow \boxed{\log_2 n = k}$$

Replacing the value of k, we get :

$$T(n) = 2cn^2\left(1-\frac{1}{2^{\log_2 n}}\right) + 2^{\log_2 n}.c$$

$$= 2cn^2\left(1-\frac{1}{n}\right) + nc$$

$$= 2cn^2 - 2cn + nc$$

$$\boxed{T(n) = O(n^2).}$$

3.) $T(n) = \begin{cases} T(\frac{n}{2}) + c\log n, & n \geq 2 \\ c, & n = 1 \end{cases}$

A: $T(n) = T(\frac{n}{2}) + c\log_2 n$

$= \left[T(\frac{n}{4}) + c\log_2 \frac{n}{2}\right] + c\log_2 n$

$= T(\frac{n}{4}) + c\left[\log_2 n + \log_2 \frac{n}{2}\right]$

$= \left[T(\frac{n}{8}) + c\log_2 \frac{n}{4}\right] + c\left[\log_2 n + \log_2 \frac{n}{2}\right]$

$= T(\frac{n}{8}) + c\left[\log_2 n + \log_2 \frac{n}{2} + \log_2 \frac{n}{4}\right]$

$= T(\frac{n}{2^3}) + c\left[\log_2 \frac{n}{2^0} + \log_2 \frac{n}{2^1} + \log_2 \frac{n}{2^2}\right]$

Let's assume that n is in the form of 2^k.

$\Rightarrow T(n) = T(\frac{n}{2^4}) + c\left[\log_2 \frac{n}{2^0} + \log_2 \frac{n}{2^1} + \log_2 \frac{n}{2^2} + \log_2 \frac{n}{2^3}\right]$

$\vdots$

$= T(\frac{n}{2^k}) + c\left[\log_2(\frac{n}{2^0}) + \log_2(\frac{n}{2^1}) + \log_2(\frac{n}{2^2}) + \cdots \log_2(\frac{n}{2^{k-1}})\right]$

Since, $n = 2^k$,

$\Rightarrow T(n) = c\left[\log_2(\frac{n}{2^0}) + \log_2(\frac{n}{2^1}) + \log_2(\frac{n}{2^2}) + \cdots \log_2(\frac{n}{2^{k-1}})\right] + T(\frac{2^k}{2^k})$

$= c\left[\log_2\left((\frac{n}{2^0})(\frac{n}{2^1})(\frac{n}{2^2}) \cdots (\frac{n}{2^{k-3}})(\frac{n}{2^{k-2}})(\frac{n}{2^{k-1}})\right)\right] + T(1)$

[k terms]

$\left[\because \log a + \log b = \log(ab)\right]$

$\Rightarrow T(n) = c\left[\log_2\left(\frac{n^k}{2^{(k-1)(\frac{k}{2})}}\right)\right] + c$

$= c\left[\log_2(n^k) - \log_2\left(2^{(k-1)(\frac{k}{2})}\right)\right] + c$

$\left[\because \log(\frac{a}{b}) = \log a - \log b\right]$

$\left[\because 2^0 \cdot 2^{k-1} = 2^1 \cdot 2^{k-2} \cdots = 2^{(\frac{k}{2})(\frac{k-2}{2})} = 2^{k-1}.\right.$ Since there are k terms and we have $(\frac{k}{2})$ pairs, total product is $\left.2^{(k-1)(\frac{k}{2})}\right]$

$$\Rightarrow T(n) = c\left[\log_2(n^k) - \log_2\left(2^{(k-1)\left(\frac{k}{2}\right)}\right)\right] + c$$

$$= c\left[k \cdot \log_2 n - (k-1)\left(\frac{k}{2}\right)\log_2 2\right] + c$$

$$= c\left[k \cdot \log_2 n - (k-1)\left(\frac{k}{2}\right)\right] + c$$

$$\left[\because \log_b a^n = n \log_b a;\ \log_a a = 1\right]$$

Also, $n = 2^k$

$$\Rightarrow \log_2 n = k \cdot \log_2 2$$

$$\Rightarrow \boxed{k = \log_2 n}$$

$$\Rightarrow T(n) = c\left[(\log_2 n)(\log_2 n) - (\log_2 n - 1)\left(\frac{\log_2 n}{2}\right)\right] + c$$

$$= c\left[(\log_2 n)^2 - \left(\frac{(\log_2 n)^2}{2} - \frac{\log_2 n}{2}\right)\right] + c$$

$$= c\left[(\log_2 n)^2 - \frac{(\log_2 n)^2}{2} + \frac{\log_2 n}{2}\right] + c$$

$$= c\left[\frac{1}{2}(\log_2 n)^2 + \frac{1}{2}(\log_2 n)\right] + c$$

$$= \frac{c}{2}\left[(\log_2 n)^2 + (\log_2 n)\right] + c$$

$$= O((\log n)^2).$$

$$\therefore \boxed{T(n) = O(\log^2 n).}$$

4) $T(n) = \begin{cases} 4T\left(\frac{n}{2}\right) + c\log n, & n \geq 2 \\ c, & n = 1 \end{cases}$

A: $T(n) = 4T\left(\frac{n}{2}\right) + c\log n$

$= 4\left[4T\left(\frac{n}{2^2}\right) + c\log\frac{n}{2}\right] + c\log n$

$= 4^2T\left(\frac{n}{2^2}\right) + c\log n + 4c\log\frac{n}{2}$

$= 4^2T\left(\frac{n}{2^2}\right) + c\left[\log n + 4\log\frac{n}{2}\right]$

$= 4^2\left[4T\left(\frac{n}{2^3}\right) + c\log\frac{n}{2^2}\right] + c\left[\log n + 4\log\frac{n}{2}\right]$

$= 4^3T\left(\frac{n}{2^3}\right) + c\left[\log n + 4\log\frac{n}{2} + 4^2\log\frac{n}{2^2}\right]$

Let's assume that n is in the form of 2^k.

$\Rightarrow T(n) = 4^4T\left(\frac{n}{2^4}\right) + c\left[\log n + 4\log\frac{n}{2} + 4^2\log\frac{n}{2^2} + 4^3\log\frac{n}{2^3}\right]$

$\vdots$

$4^kT\left(\frac{n}{2^k}\right) + c\left[\log n + 4\log\frac{n}{2} + \dots 4^{k-1}\log\frac{n}{2^{k-1}}\right]$

Since, $\boxed{n = 2^k}$

$\Rightarrow T(n) = c\left[\log n + 4\log\frac{n}{2} + 4^2\log\frac{n}{2^2} + \dots 4^{k-1}\log\frac{n}{2^{k-1}}\right] + 4^kT\left(\frac{2^k}{2^k}\right)$

$= c\left[\log n + \log\left(\frac{n}{2}\right)^4 + \log\left(\frac{n}{2^2}\right)^{4^2} + \dots \log\left(\frac{n}{2^{k-1}}\right)^{4^{k-1}}\right] + 4^k \cdot T(1)$

$[\because n\log a = \log a^n]$

$= c\left[\log\left(\frac{n \times n^4 \times n^{4^2} \times n^{4^3} \times \dots\dots n^{4^{k-1}}}{2^4 \times (2^2)^{4^2} \times (2^3)^{4^3} \times \dots\dots (2^{k-1})^{4^{k-1}}}\right)\right] + 4^k$

$[\because \log a + \log b = \log(ab)]$

$= c\left[\log\left(\frac{n^{(1+4+4^2+4^3+\dots 4^{k-1})}}{2^{(4+2\cdot4^2+3\cdot4^3+\dots(k-1)4^{k-1})}}\right)\right] + 4^k$

$$= c\left[\log\left(\frac{n^{(4^0+4^1+4^2+4^3+\cdots 4^{k-1})}}{2^{(1\cdot 4^1+2\cdot 4^2+3\cdot 4^3+4\cdot 4^4+\cdots (k-1)4^{k-1})}}\right)\right] + 4^k$$

[By using Infinite G.P and A.G.P formula,

$$\sum_{i=0}^{\infty} r^i = \frac{a}{1-r} \; ; \; \sum_{i=0}^{\infty} i.r^i = \frac{a}{(1-r)} + \frac{dr}{(1-r)^2}$$]

Since, we need to find the upper bound, infinite sum formula can be used.

$$\Rightarrow T(n) = c\left[\log\left(\frac{n^{\frac{1}{(1-4)}}}{2^{\left(\frac{1}{1-4}+\frac{(1)(4)}{(-3)^2}\right)}}\right)\right] + 4^k$$

$$= c\left[\log\left(\frac{n^{\frac{-1}{3}}}{2^{\left(\frac{-1}{3}+\frac{4}{9}\right)}}\right)\right] + 4^k$$

$$= c\left[\log_2\left(\frac{n^{\frac{-1}{3}}}{2^{\frac{2}{9}}}\right)\right] + 4^k$$

$$= c\left[\log_2\left(n^{\frac{-1}{3}}\right) - \log_2\left(2^{\frac{2}{9}}\right)\right] + 4^k$$

$$= c\left[\frac{-1}{3}\log_2 n - \frac{2}{9}\log_2 2\right] + 4^k$$

Also, $n = 2^k$

$$\Rightarrow \log_2 n = k.\log_2 2$$

$$\Rightarrow \boxed{k = \log_2 n}$$

$$\Rightarrow T(n) = c\left[\frac{-1}{3}\log n - \frac{2}{9}\right] + 4^{\log_2 n}$$

$$= c\left[\frac{-1}{3}\log n - \frac{2}{9}\right] + 2^{\log_2 n^2}$$

$$= c\left[\frac{-1}{3}\log n - \frac{2}{9}\right] + n^2 \quad \left[\because a^{\log_a n} = n\right]$$

$$\boxed{T(n) = O(n^2).}$$

5.) $T(n) = \begin{cases} 3T\left(\frac{n}{2}\right) + cn & , n \geq 2 \\ c & , n = 1 \end{cases}$

A: $T(n) = 3T\left(\frac{n}{2}\right) + cn$

$= 3\left[3T\left(\frac{n}{4}\right) + c\frac{n}{2}\right] + cn$

$= 9T\left(\frac{n}{4}\right) + cn\left[1 + \frac{3}{2}\right]$

$= 3^2T\left(\frac{n}{2^2}\right) + cn\left[1 + \frac{3}{2}\right]$

$= 3^3T\left(\frac{n}{2^3}\right) + cn\left[1 + \frac{3}{2} + \frac{3^2}{2^2}\right]$

$= 3^4T\left(\frac{n}{2^4}\right) + cn\left[\frac{3^0}{2^0} + \frac{3^1}{2^1} + \frac{3^2}{2^2} + \frac{3^3}{2^3}\right]$

$\vdots$

$= 3^kT\left(\frac{n}{2^k}\right) + cn\left[\frac{3^0}{2^0} + \frac{3^1}{2^1} + \frac{3^2}{2^2} + \cdots \frac{3^{k-1}}{2^{k-1}}\right]$

Lets assume that n is in the form of 2^k.

$\Rightarrow n = 2^k$

$\Rightarrow T(n) = 3^kT\left(\frac{2^k}{2^k}\right) + cn\left[\frac{3^0}{2^0} + \frac{3^1}{2^1} + \frac{3^2}{2^2} + \cdots \frac{3^{k-1}}{2^{k-1}}\right]$

$= 3^k T(1) + cn\left[\frac{3^0}{2^0} + \frac{3^1}{2^1} + \frac{3^2}{2^2} + \cdots \frac{3^{k-1}}{2^{k-1}}\right]$

$= 3^k \cdot c + cn\left[\frac{\left(\frac{3}{2}\right)^k - 1}{\left(\frac{3}{2} - 1\right)}\right]$ $\left[\because T(1) = c \text{ \& } \sum_{i=0}^{k-1} r^i = \frac{(a)(r^k - 1)}{(r-1)}\right]$

$= 3^k \cdot c + cn\left[\frac{\left(\frac{3}{2}\right)^k - 1}{\left(\frac{1}{2}\right)}\right]$

$= 3^k \cdot c + 2cn\left[\left(\frac{3}{2}\right)^k - 1\right]$

Also, $n = 2^k$

$\Rightarrow \boxed{\log n = k}$

$$\Rightarrow T(n) = c.3^{\log_2 n} + 2cn\left[\left(\frac{3}{2}\right)^{\log_2 n} - 1\right]$$

$$= c.n^{\log_2 3} + 2cn\left[n^{\log_2 \left(\frac{3}{2}\right)} - 1\right] \quad \left[\because a^{\log_b c} = c^{\log_b a}\right]$$

$$= c.n^{\log_2 3} + 2cn\left[n^{(\log_2 3 - \log_2 2)} - 1\right]$$

$$= c.n^{\log_2 3} + 2cn\left[n^{(\log_2 3 - 1)} - 1\right]$$

$$= cn\log_2 3 + 2c\left[n^{(\log_2 3 \cancel{-1+1})} - n\right]$$

$$= c.n \log_2 3 + 2c.n^{\log_2 3} - 2cn$$

$$\boxed{T(n) = O(n^{\lg 3}).}$$

6) $T(n) = \begin{cases} 7T(\frac{n}{2}) + cn^2 & , n \geq 2 \\ c & , n \leq 2 \end{cases}$

A: $T(n) = 7T(\frac{n}{2}) + cn^2$

$= 7\left[7T(\frac{n}{4}) + c(\frac{n}{2})^2\right] + cn^2$

$= 7^2 T(\frac{n}{2^2}) + cn^2 + 7c(\frac{n}{2})^2$

$= 7^2 T(\frac{n}{2^2}) + cn^2\left[1 + 7(\frac{1}{2})^2\right]$

$= 7^2\left[7T(\frac{n}{2^3}) + cn^2(\frac{1}{2})^2\right] + cn^2\left[1 + 7(\frac{1}{2})^2\right]$

$= 7^3 T(\frac{n}{2^3}) + cn^2\left[1 + (7)(\frac{1}{2})^2 + (7^2)(\frac{1}{2^2})^2\right]$

$\vdots$

$= 7^k T(\frac{n}{2^k}) + cn^2\left[1 + 7^1 \cdot (\frac{1}{2^1})^2 + 7^2(\frac{1}{2^2})^2 + \cdots 7^{k-1}(\frac{1}{2^{k-1}})^2\right]$

Lets assume that n is in the form of 2^k.

$\Rightarrow T(n) = 7^k \cdot T(\frac{2^k}{2^k}) + cn^2\left[\sum_{i=0}^{k-1} \frac{7^i}{2^{2i}}\right]$ [∵ GP Series with $a=1$, $r = \frac{7}{4}$]

$\Rightarrow T(n) = 7^k \cdot c + cn^2\left[\sum_{i=0}^{k-1} (\frac{7}{4})^i\right]$ [∵ $n = 2^k \Rightarrow k = \log_2 n$]

$= c \cdot 7^{\log_2 n} + cn^2\left[\frac{(\frac{7}{4})^{\log_2 n} - 1}{(\frac{7}{4} - 1)}\right]$

$= c \cdot 7^{\log_2 n} + cn^2\left[\frac{(\frac{7}{4})^{\log_2 n} - 1}{(\frac{3}{4})}\right]$

$= c \cdot 7^{\log_2 n} + \frac{4}{3}cn^2\left[\frac{7^{\log_2 n}}{4^{\log_2 n}} - 1\right]$

$$\Rightarrow T(n) = c \cdot 7^{\log_2 n} + \frac{4}{3} cn^2 \left[\frac{7^{\log_2 n}}{2^{\log_2 n^2}} - 1 \right]$$

$$= c \cdot 7^{\log_2 n} + \frac{4}{3} cn^2 \left[\frac{7^{\log_2 n}}{n^2} - 1 \right]$$

$$= c \cdot 7^{\log_2 n} + \frac{4}{3} c \cdot (7^{\log_2 n}) - \frac{4}{3} cn^2$$

$$= \frac{7}{3} c \cdot 7^{\log_2 n} - \frac{4}{3} cn^2$$

$$= \frac{7}{3} c \cdot n^{\log_2 7} - \frac{4}{3} cn^2 \quad \left[\because a^{\log_b c} = c^{\log_b a} \right]$$

$$= \frac{7}{3} c \cdot n^{2.8} - \frac{4}{3} cn^2$$

$$\boxed{T(n) = O(n^{\log_2 7}).}$$

7) $T(n) = \begin{cases} 2T(\frac{n}{2}) + cn\log n, & n \geq 2 \\ c, & n = 1 \end{cases}$

A: $T(n) = 2T(\frac{n}{2}) + cn\log n$

$= 2[2T(\frac{n}{4}) + c(\frac{n}{2})\log(\frac{n}{2})] + cn\log n$

$= 2^2T(\frac{n}{2^2}) + 2c(\frac{n}{2})\log(\frac{n}{2}) + cn\log n$

$= 2^2T(\frac{n}{2^2}) + cn[\log n + \log\frac{n}{2}]$

$= 2^2[2T(\frac{n}{2^3}) + c(\frac{n}{4})\log(\frac{n}{4})] + cn[\log n + \log\frac{n}{2}]$

$= 2^3T(\frac{n}{2^3}) + cn[\log n + \log\frac{n}{2} + \log\frac{n}{4}]$

Let's assume that n is of the form, 2^k.

$\Rightarrow T(n) = 2^kT(\frac{n}{2^k}) + cn[\log n + \log\frac{n}{2} + \log\frac{n}{4} + \cdots \log\frac{n}{2^{k-1}}]$

$\because \boxed{n = 2^k}$

$\Rightarrow T(n) = 2^kT(\frac{2^k}{2^k}) + cn[\log n + \log\frac{n}{2} + \log\frac{n}{2^2} + \cdots \log\frac{n}{2^{k-1}}]$

$= 2^k \cdot T(1) + cn[\log_2(\frac{n \times n \times n \cdots \times n}{2^0 \times 2^1 \times 2^2 \times \cdots \times 2^{k-1}})]$

[$\because \log a + \log b = \log(ab)$]

$= 2^k \cdot c + cn[\log_2(\frac{n^k}{2^{k(\frac{k-1}{2})}})]$ [$\because$ there are k terms and $\frac{k}{2}$ pairs equal to 2^{k-1}.

$= 2^k \cdot c + cn[\log(n^k) - \log_2(2^{(\frac{k}{2})(k-1)})]$ $T(1) = c$]

[$\because \log(\frac{a}{b}) = \log a - \log b$]

$= 2^k \cdot c + cn[k \cdot \log_2 n - (\frac{k}{2})(k-1)\log_2 2]$

$= 2^k \cdot c + cn[k \cdot \log_2 n - (\frac{k}{2})(k-1)]$ [$\because \log_a a = 1$]

$$T(n) = 2^k \cdot c + cnk\left[\log_2 n - \frac{(k-1)}{2}\right]$$

$$= 2^k \cdot c + cnk\left[\log_2 n - \frac{k}{2} + \frac{1}{2}\right]$$

Also, $n = 2^k$

$$\Rightarrow \boxed{\log_2 n = k}$$

$$\Rightarrow T(n) = 2^{\log_2 n} \cdot c + cn\log_2 n\left[\log_2 n - \frac{\log_2 n}{2} + \frac{1}{2}\right]$$

$$= 2^{\log_2 n} \cdot c + cn\log_2 n\left[\frac{1}{2}\log_2 n + \frac{1}{2}\right]$$

$$= 2^{\log_2 n} \cdot c + \frac{cn}{2} \cdot (\log_2 n)^2 + \frac{cn}{2}\log_2 n$$

$$= cn\left[\frac{2 + (\log_2 n)^2 + \log_2 n}{2}\right]$$

$$= \frac{2cn + cn(\log_2 n)^2 + cn\log_2 n}{2}$$

$$\boxed{T(n) = O(n \cdot \log^2 n)}$$

8.) $T(n) = \begin{cases} T(\frac{n}{4}) + cn\log n, & n \geq 2 \\ c, & n = 1 \end{cases}$

A: $T(n) = T(\frac{n}{4}) + cn\log n$

$= T(\frac{n}{4^2}) + c(\frac{n}{4})\log\frac{n}{4} + cn\log n$

$= T(\frac{n}{4^2}) + cn[\log n + \frac{1}{4}\log\frac{n}{4}]$

$= T(\frac{n}{4^3}) + cn[\log n + \frac{1}{4}\log\frac{n}{4} + \frac{1}{4^2}\log\frac{n}{4^2}]$

Lets assume that n is of the form, 4^k.

$\Rightarrow T(n) = T(\frac{n}{4^4}) + cn[\log n + \frac{1}{4}\log\frac{n}{4} + \frac{1}{4^2}\log\frac{n}{4^2} + \frac{1}{4^3}\log\frac{n}{4^3}]$

$= T(\frac{n}{4^k}) + cn[\log n + \frac{1}{4}\log\frac{n}{4} + \cdots \frac{1}{4^{k-1}}\log\frac{n}{4^{k-1}}]$

Since, $\boxed{n = 4^k}$

$\Rightarrow T(n) = T(\frac{4^k}{4^k}) + cn[\log_4 n + \log_4(\frac{n}{4})^{\frac{1}{4}} + \log_4(\frac{n}{4^2})^{\frac{1}{4^2}} + \cdots \log_4(\frac{n}{4^{k-1}})^{\frac{1}{4^{k-1}}}]$

$[\because n\log a = \log a^n]$

$= T(1) + cn[\log_4(n \times (\frac{n}{4})^{\frac{1}{4}} \times (\frac{n}{4^2})^{\frac{1}{4^2}} \cdots (\frac{n}{4^{k-1}})^{\frac{1}{4^{k-1}}})]$

$[\because \log a + \log b = \log ab]$

$= c + cn[\log_4(\frac{n^{1+\frac{1}{4}+\frac{1}{4^2}+\cdots\frac{1}{4^{k-1}}}}{4^{\frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\cdots\frac{k-1}{4^{k-1}}}})]$

[Let $S = \sum_{m=0}^{\infty}\frac{m}{4^m} = \frac{1}{4} + \frac{2}{4^2} + \frac{3}{4^3} + \cdots \infty$ terms

$= \frac{1}{4^1} + (\frac{1}{4^2} + \frac{1}{4^2}) + (\frac{1}{4^3} + \frac{2}{4^3}) + \cdots$

$= (\frac{1}{4^1} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots) + (\frac{1}{4^2} + \frac{2}{4^3} + \cdots)$

$= (\frac{1}{4^1} + \frac{1}{4^2} + \cdots) + \frac{1}{4}(\frac{1}{4^1} + \frac{2}{4^2} + \cdots)$

$S = (\frac{1}{4^1} + \frac{1}{4^2} + \cdots) + \frac{1}{4}(S)$

By Infinite GP Sum ; $\sum_{m=1}^{\infty} \frac{1}{4^m} = \frac{a}{1-r} = \frac{\frac{1}{4}}{\left(1-\frac{1}{4}\right)}$

$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$

$\Rightarrow S = \frac{1}{3} + \frac{S}{4} \Rightarrow S - \frac{S}{4} = \frac{1}{3}$

$\Rightarrow \frac{3S}{4} = \frac{1}{3} \Rightarrow \boxed{S = \frac{4}{9}}$]

$\Rightarrow T(n) = c + cn\left[\log_4\left(\frac{n^{\left(\frac{1}{1-\frac{1}{4}}\right)}}{4^{\left(\frac{4}{9}\right)}}\right)\right]$

$= c + cn\left[\log_4\left(\frac{n^{\frac{4}{3}}}{4^{\frac{4}{9}}}\right)\right]$ $\left[\because \log\left(\frac{a}{b}\right) = \log a - \log b\right]$

$= c + cn\left[\log_4 n^{\frac{4}{3}} - \log_4 4^{\frac{4}{9}}\right]$

$= c + cn\left[\frac{4}{3}\log_4 n - \frac{4}{9}\right]$ $\left[\because \log a^n = n\log a;\ \log_a a = 1\right]$

$= c + \frac{4}{3}cn\log_4 n - \frac{4}{9}cn$

$\boxed{T(n) = O(n\log n).}$

9.) $T(n) = \begin{cases} 64\,T\left(\frac{n}{2}\right) + n^{\frac{1}{2}}, & n \geq 2 \\ c, & n = 1 \end{cases}$

A: $T(n) = 64T\left(\frac{n}{2}\right) + n^{\frac{1}{2}}$

$= 64\left[64\,T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^{\frac{1}{2}}\right] + n^{\frac{1}{2}}$

$= 64^2 T\left(\frac{n}{4}\right) + n^{\frac{1}{2}}\left[1 + 64\left(\frac{1}{2}\right)^{\frac{1}{2}}\right]$

$= 64^3 T\left(\frac{n}{2^3}\right) + n^{\frac{1}{2}}\left[1 + 64\left(\frac{1}{2}\right)^{\frac{1}{2}} + 64^2\left(\frac{1}{2^2}\right)^{\frac{1}{2}}\right]$

Lets assume that n is of the form 2^k.

$\Rightarrow T(n) = 64^k\,T\left(\frac{n}{2^k}\right) + n^{\frac{1}{2}}\left[1 + \left(\frac{64}{\sqrt{2}}\right)^1 + \left(\frac{64}{\sqrt{2}}\right)^2 + \cdots \left(\frac{64}{\sqrt{2}}\right)^{k-1}\right]$

$\because \boxed{n = 2^k}$

$\Rightarrow T(n) = 64^k \cdot T\left(\frac{2^k}{2^k}\right) + \sqrt{n}\left[1 + \left(\frac{64}{\sqrt{2}}\right)^1 + \left(\frac{64}{\sqrt{2}}\right)^2 + \cdots \left(\frac{64}{\sqrt{2}}\right)^{k-1}\right]$

$= 64^k \cdot T(1) + \sqrt{n}\left[\frac{(1)\left[\left(\frac{64}{\sqrt{2}}\right)^k - 1\right]}{\left(\frac{64}{\sqrt{2}} - 1\right)}\right]$

$\left[\because \text{GP Series sum} = a\left(\frac{r^n - 1}{r - 1}\right)\right]$

$= 64^k \cdot c + \frac{\sqrt{n}}{32\sqrt{2} - 1}\left[\left(\frac{64}{\sqrt{2}}\right)^k - 1\right]$

Since, $n = 2^k$

$\Rightarrow \boxed{\log_2 n = k}$

$\Rightarrow T(n) = 64^{\log_2 n} \cdot c + \frac{\sqrt{n}}{32\sqrt{2} - 1}\left[(32\sqrt{2})^{\log_2 n} - 1\right]$

$$\Rightarrow T(n) = c \cdot 2^{6 \cdot \log_2 n} + \frac{\sqrt{n}}{32\sqrt{2}-1}\left[(\sqrt{2})^{11 \log_{(\sqrt{2})^2} n} - 1 \right]$$

$$= c \cdot 2^{\log_2 n^6} + \frac{\sqrt{n}}{32\sqrt{2}-1}\left[(\sqrt{2})^{\log_{(\sqrt{2})} n^{\frac{11}{2}}} - 1 \right]$$

$$= c \cdot n^6 + \frac{\sqrt{n}}{32\sqrt{2}-1}\left[n^{\frac{11}{2}} - 1 \right]$$

$$\left[\because n \log a = \log a^n ; \quad n \log_{b^m} a = \frac{n}{m} \log_b a ; \quad = \log_b a^{\left(\frac{n}{m}\right)} \right]$$

$$\Rightarrow T(n) = c \cdot n^6 + \frac{n^{\frac{11}{2}+\frac{1}{2}} - n^{\frac{1}{2}}}{32\sqrt{2}-1}$$

$$= c \cdot n^6 + \frac{n^6 - \sqrt{n}}{32\sqrt{2}-1}$$

$$= n^6\left[c + \frac{1}{32\sqrt{2}-1} \right] - \frac{\sqrt{n}}{32\sqrt{2}-1}$$

$$\boxed{T(n) = O(n^6).}$$

10.) $T(n) = \begin{cases} 32T\left(\frac{n}{2}\right) + n^2 \log n, & n \geq 2 \\ c, & n = 1 \end{cases}$

A: $T(n) = 32T\left(\frac{n}{2}\right) + n^2 \log n$

$$= 32\left[32T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2 \log \frac{n}{2}\right] + n^2 \log n$$

$$= 32^2 T\left(\frac{n}{2^2}\right) + n^2\left[\log n + 32\left(\frac{1}{2^1}\right)^2 \log \frac{n}{2}\right]$$

$$= 32^3 T\left(\frac{n}{2^3}\right) + n^2\left[\log n + 8 \log \frac{n}{2} + 8^2 \log \frac{n}{2^2}\right]$$

Lets assume that n is of the form, 2^k.

$$\Rightarrow T(n) = 32^k T\left(\frac{n}{2^k}\right) + n^2\left[\log n + 8 \log \frac{n}{2} + \cdots 8^{k-1} \log\left(\frac{n}{2^{k-1}}\right)\right]$$

Since, $\boxed{n = 2^k}$

$[\because n \log a = \log a^n]$

$$\Rightarrow T(n) = 32^k T\left(\frac{2^k}{2^k}\right) + n^2\left[\log n + \log\left(\frac{n}{2}\right)^8 + \log\left(\frac{n}{2^2}\right)^{8^2} + \cdots \log\left(\frac{n}{2^{k-1}}\right)^{8^{k-1}}\right]$$

$$= 32^k \cdot T(1) + n^2\left[\log\left(n \cdot \left(\frac{n}{2}\right)^8 \cdot \left(\frac{n}{2^2}\right)^{8^2} \cdots \left(\frac{n}{2^{k-1}}\right)^{8^{k-1}}\right)\right]$$

$$= 32^k \cdot c + n^2\left[\log\left(\frac{n^{1+8^1+8^2+\cdots 8^{k-1}}}{2^{1 \cdot 8^1 + 2 \cdot 8^2 + \cdots (k-1) \cdot 8^{k-1}}}\right)\right]$$

$[\because T(1) = c;\ \log a + \log b = \log ab]$

$$= 32^k \cdot c + n^2\left[\log\left(\frac{\sum_{i=0}^{\infty} n^{8^i}}{\sum_{j=0}^{\infty} 2^{k \cdot 8^k}}\right)\right] \quad \left[\because \sum_{i=0}^{\infty} r^i = \frac{a}{1-r};\right.$$

$$= 32^k \cdot c + n^2\left[\log\left(\frac{n^{\left(\frac{1}{1-8}\right)}}{2^{\left(\frac{1}{-7} + \frac{8}{(-7)^2}\right)}}\right)\right] \quad \left.\sum_{i=0}^{\infty} i \cdot r^i = \frac{a}{1-r} + \frac{dr}{(1-r)^2}\right]$$

$$= 32^k \cdot c + n^2 \left[\log\left(\frac{n^{\frac{-1}{7}}}{2^{\frac{1}{49}}} \right) \right] \quad \left[\because \log\left(\frac{a}{b}\right) = \log a - \log b \right]$$

$$= 32^k \cdot c + n^2 \left[\log\left(n^{\frac{-1}{7}}\right) - \log\left(2^{\frac{1}{49}}\right) \right]$$

Also, $n = 2^k$

$$\Rightarrow \boxed{k = \log_2 n}$$

$$\Rightarrow T(n) = 32^{\log_2 n} + n^2 \left[\frac{-1}{7} \log_2 n - \frac{1}{49} \log_2 2 \right]$$

$$= n^5 + n^2 \left[\frac{-1}{7} \log n - \frac{1}{49} \right]$$

$$\left[\because (32)^{\log_2 n} = (2^5)^{\log_2 n} = 2^{\log_2 n^5} = n^5;\ \log_2 2 = 1 \right]$$

$$\Rightarrow T(n) = n^5 + (-n^2) \left[\frac{\log n}{7} + \frac{1}{49} \right]$$

$$= n^5 - \frac{n^2 \log n}{7} - \frac{n^2}{49}.$$

$$\boxed{T(n) = O(n^5).}$$

iii) $T(n) = \begin{cases} 4T(\frac{n}{2}) + cn & ; n \geq 2 \\ c & , n = 1 \end{cases}$

A: $T(n) = 4T(\frac{n}{2}) + cn$

$= 4\left[4T(\frac{n}{2^2}) + c(\frac{n}{2})\right] + cn$

$= 4^2T(\frac{n}{2^2}) + cn\left[1 + \frac{4}{2}\right]$

$= 4^3T(\frac{n}{2^3}) + cn\left[1 + \frac{4}{2} + \frac{16}{4}\right]$

$= 4^4T(\frac{n}{2^4}) + cn\left[1 + 2 + 2^2 + 2^3\right]$

Lets assume that n is of the form, 2^k.

$\Rightarrow T(n) = 4^k \cdot T(\frac{n}{2^k}) + cn\left[1 + 2 + 2^2 + \cdots 2^{k-1}\right]$

Since, $\boxed{n = 2^k}$

$\Rightarrow T(n) = 4^k \cdot T(\frac{2^k}{2^k}) + cn\left[\frac{(1)(2^k - 1)}{(2-1)}\right]$

$\left[\text{GP sum} = \frac{(a)(r^n - 1)}{(r-1)}\right]$

$\Rightarrow T(n) = 2^{2k} \cdot T(1) + cn(2^k - 1)$

Since, $n = 2^k$

$\Rightarrow \boxed{\log_2 n = k}$

$\Rightarrow T(n) = 2^{\log_2 n^2} \cdot c + cn(2^{\log_2 n} - 1)$

$= c \cdot n^2 + cn(n-1)$

$= 2cn^2 - cn$

$T(n) = O(n^2)$

12) $T(n) = \begin{cases} 3T\left(\frac{n}{4}\right) + cn, & n \geq 2 \\ c, & n = 1 \end{cases}$

A: $T(n) = 3T\left(\frac{n}{4}\right) + cn$

$$= 3\left[3T\left(\frac{n}{4^2}\right) + \frac{cn}{4}\right] + cn$$

$$= 3^2 T\left(\frac{n}{4^2}\right) + \frac{3nc}{4} + cn$$

$$= 3^2\left[3T\left(\frac{n}{4^3}\right) + \frac{cn}{4^2}\right] + cn\left[1 + \frac{3}{4}\right]$$

$$= 3^3 T\left(\frac{n}{4^3}\right) + cn\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2\right]$$

Lets assume that n is of the form 4^K.

$$\Rightarrow T(n) = 3^K \cdot T\left(\frac{n}{4^K}\right) + cn\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots \left(\frac{3}{4}\right)^{K-1}\right]$$

Since, $n = (2^2)^K = 4^K$

$$\Rightarrow T(n) = 3^K \cdot T\left(\frac{4^K}{4^K}\right) + cn\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots \left(\frac{3}{4}\right)^{K-1}\right]$$

$\left[\because T(1) = c \text{ ; Infinite GP Sum} = \frac{a}{1-r}\right]$

$$\Rightarrow T(n) = 3^K \cdot c + cn\left[\frac{1}{\left(1 - \frac{3}{4}\right)}\right]$$

$$= 3^K \cdot c + 4cn$$

Since, $n = 4^K$

$$\Rightarrow \boxed{\log_4 n = K}$$

$$\Rightarrow T(n) = 3^{\log_4 n} \cdot c + 4cn = 4cn + n^{\log_4 3}$$

$$\boxed{T(n) = O(n).}$$

13.) $T(n) = \begin{cases} 9T(\frac{n}{3}) + n^2\log n, & n \geq 3 \\ c & ; n = 1 \end{cases}$

A: $T(n) = 9T(\frac{n}{3}) + n^2\log n$

$= 9\left[9T(\frac{n}{3^2}) + (\frac{n}{3})^2\log(\frac{n}{3})\right] + n^2\log n$

$= 9^2T(\frac{n}{3^2}) + n^2\left[\log n + \cancel{9}(\frac{1}{\cancel{3}})^2\log\frac{n}{3}\right]$

$= 9^2T(\frac{n}{3^2}) + n^2\left[\log n + \log\frac{n}{3}\right]$

$= 9^3T(\frac{n}{3^3}) + n^2\left[\log n + \log\frac{n}{3} + \log\frac{n}{3^2}\right]$

Lets assume that n is of the form, 3^k.

$\Rightarrow T(n) = 9^k \cdot T(\frac{n}{3^k}) + n^2\left[\log n + \log\frac{n}{3} + \cdots \log\frac{n}{3^{k-1}}\right]$

Since, $\boxed{n = 3^k}$

$\Rightarrow T(n) = 9^k \cdot T(\frac{3^k}{3^k}) + n^2\left[\log_3\left((n)(\frac{n}{3})(\frac{n}{3^2})\cdots(\frac{n}{3^{k-1}})\right)\right]$

$[\because \log a + \log b = \log ab]$

$\Rightarrow T(n) = 9^k \cdot T(1) + n^2\left[\log_3\left(\frac{n^k}{3^{1+2+\cdots k-1}}\right)\right]$

$= 3^{2k} \cdot c + n^2\left[\log_3\left(\frac{n^k}{3^{(k)(\frac{k-1}{2})}}\right)\right]$ [Sum of 1st n terms $= \frac{(n)(n+1)}{2}$]

$= 3^{2k} \cdot c + n^2\left[\log_3 n^k - \log_3 3^{(k)(\frac{k-1}{2})}\right]$ $[\because \log(\frac{a}{b}) = \log a - \log b]$

$= 3^{2k} \cdot c + n^2\left[k \cdot \log_3 n - (k)(\frac{k-1}{2})\right]$ $[\because \log a^n = n\log a;$

$= 3^{2k} \cdot c + n^2\left[k \cdot \log n - \frac{k^2}{2} + \frac{k}{2}\right]$ $\log_a a = 1]$

Since, $n = 3^k$

$\Rightarrow \log_3 n = k \cdot \log_3 3$

$\Rightarrow \boxed{k = \log_3 n}$

Q. $T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + 1, & n > 1 \quad — ① \\ 0, & n = 1 \end{cases}$

Sol:- Backward Substitution:-

In this method, we replace n with $n/2$ ways this will be followed for 3 steps and then will generalise the steps (K's)

So at first step:
K=1,

from ①

$$T(n) = 2.T(n/2) + 1$$

replace n with $n/2$

$$\therefore T(n/2) = 2.T\left(\frac{\frac{n}{2}}{2}\right) + 1 \quad — ②$$

Substituting ② in ①

$$T(n) = 2\times\left[2.T(n/4) + 1\right] + 1$$

$$T(n) = 4T(n/4) + 2 + 1 \quad — ③$$

Second step,
K = 2

replace $n/2$ with $n/2$ in ①

$$T(n/4) = 2.T\left(\frac{\frac{n}{4}}{2}\right) + 1$$

$$= 2.T(n/8) + 1 \quad — ④$$

Sub ③ with ④

$$T(n) = 4\times\left[2T(n/8) + 1\right] + 2 + 1$$

$$T(n) = 8.T(n/8) + 4 + 2 + 1 \quad — ⑤$$

As we can see to it that, we have been breaking the problem into 2 subproblems of half size, so by assumption that n is in the form of 2^K.

And with some constant term K_0

So by proceeding by assuming that n is of the form of 2^K.

As mentioned earlier, generalisation,

So by generalising steps ①, ③, ⑤.

$$\Rightarrow 2^K \cdot T(n/2^K) + 2^{K-1} + 2^{K-2} + 2^{K-3} \quad - ⑥$$

$$\Rightarrow 2^K \cdot T(n/2^K) + \sum_{i}^{K-1} 2^i \quad [i = 0 \ldots K-1]$$

here we have taken summation process.

Summation property

So by using formula for sum of terms in G.P

$$a + ar + ar^2 + \ldots + ar^{n-1}$$

$$\Rightarrow \frac{a(1-r^n)}{r-1} \quad [r<1]$$

$$\text{Summation result} \Rightarrow \frac{2(r^n-1)}{r-1}$$

$$\Rightarrow \frac{2(2^{K-1}-1)}{2-1}$$

So result $\Rightarrow \underline{\underline{2^K - 2}}$

As mentioned in, T(1) is the best case

And also $n = 2^K$ ∴ Applying logarithm for simplication;

$\log_2 n = \log_2 (2^k)$

$\boxed{\log_2 n = k}$

$\therefore$ Replacing the value of k in 6

$T(n) = 2^{\log_2 n} \times T\left(\frac{n}{2^{\log_2 n}}\right) + 2^{\log_2 n} - 2$

$\Rightarrow n.T\left(\frac{n}{n}\right) + n - 2 \quad \left[2^{\log_2 n} = n \right.$

$\Rightarrow n.T(1) + n - 2$

$\Rightarrow \underline{n-2} \quad \left[\text{From base case } T(1) = 0 \right.$

$\underline{\underline{T(n) = \Theta(n)}}$

Q. $T(n) = 2T(n-1) + 1, \; n > 0 \quad - 1$

This relation is of from Tower of Hanoi

Sol:- Backward substitution

In this problem, we replace n with (n-1), (n-2) respectively in various steps to obtain a general expression

So at first step,

from 1

$T(n) = 2T(n-1) + 1$

$T(n-1) = 2T(n-2) + 1 \quad - 2$

Sub 2 in 1

$T(n) = 2\left[2T(n-2) + 1\right] + 1$

$T(n) = 4T(n-2) + 2 + 1$ — ③

So proceeding with next step;
replacing with n-2 in ①

$T(n-2) = 2T(n-3) + 1$

sub in ③

$T(n) = 4[2T(n-3)+1] + 2 + 1$

$T(n) = 8T(n-3) + 4 + 2 + 1$ — ④

As we are breaking our problem into two subproblem $\therefore n = 2^k$ [assumption]

As mentioned, the generalised expression will be

from ①, ③ & ④ we get ⑤

$$\Rightarrow 2^k \cdot T(n-k) + \underbrace{2^{k-1} + 2^{k-2} + 2^{k-3}}_{k \text{ times}}$$

So taking summation, using sum of terms in G.P

$$\sum_{i}^{k-1} 2^i \Rightarrow \frac{2(2^{k-1}-1)}{2-1} \quad \left[\frac{a(r^n-1)}{r-1} \quad \begin{array}{l} a = r = 2 \\ n = k-1 \end{array}\right.$$

$$= \frac{2 \times 2^{k-1} - 2}{1} \Rightarrow \underline{\underline{2^k - 2}}$$

As mentioned $T(1)$ is best case and equals to 0

$T(n-k) = 0$ (best case)

$\therefore \boxed{n = k}$

So replacing the value of k in (5)

$$T(n) = 2^n \cdot T(n-n) + 2^n - 2$$

$$\Rightarrow 2^n \times 0 + 2^n - 2$$

$$\therefore T(n) = O(2^n)$$

Q. $T(n) = 2T(n/2) + n/\log n$ — (1)

Sol:- Backward Substitution

In this problem, we replace n with $n/2$, $n/2^2$ respectively in various step to obtain a general expression

So at first step

from (1)

replace n with $n/2$

$$T(n/2) = 2T\left(\frac{n/2}{2}\right) + \frac{n/2}{\log n/2} \quad -(2)$$

Next step

replace with $n/4$

$$T(n/2^2) = 2T(n/2^3) + \frac{n/2^2}{\log n/2^2} \quad -(3)$$

Substituting (2), (3) in (1) for $T(n)$

$$\therefore T(n) = 2\left[2T(n/2^2) + \frac{n/2}{\log n/2}\right] + n/\log n$$

$$= 2^2 \cdot T(n/2^2) + \frac{n}{\log n/2} + n/\log n$$

$$\Rightarrow 2^2 \cdot \left[2T(n/2^3) + \frac{n/2^2}{\log n/2^2}\right] + n/\log n/2 + n/\log n$$

$$\Rightarrow 2^3 \cdot T(n/2^3) + \frac{n}{\log n/2^2} + \frac{n}{\log n/2} + \frac{n}{\log n} \quad — (4)$$

Next process, will be same as we have done in previous problems, assuming $n = 2^k$, as diving the problem into two subproblem of half sizes.

As mentioned, the general expression obtained from (1), (3) & (4)

$$\Rightarrow 2^k \cdot T(n/2^k) + n\underbrace{\left[\frac{1}{\log n/2^{k-1}} + \cdots + \frac{1}{\log n/2^{k-k}}\right]}_{k \text{ times}} \quad — (5)$$

So taking the sum of terms (G.P) method.

$$\sum_{i=0}^{k-1} \frac{1}{\log n/2^i} \Rightarrow \frac{1}{\log \frac{2^k}{2^i}} \quad \left[n = 2^k \right.$$

where, $\log \frac{2^k}{2^i} \Rightarrow \log 2^k - \log 2^i$

$\Rightarrow k - i \quad \left[\log 2^k \Rightarrow k \right.$

$$\therefore \sum_{i=0}^{k-1} \frac{1}{k-i}$$

By taking summation, here k is constant,

$$\therefore \sum_{i=1}^{k} \frac{1}{i} = \log k \quad \left[\text{Summation property} \right.$$

$$\therefore \Rightarrow \sum_{i=0}^{k-1} \frac{1}{k-i} \Rightarrow \underline{\underline{\log . \log k}}$$

$\Rightarrow$ Sub this in (5)

$$\Rightarrow 2^k + n \times \log . \log n \quad \left[\begin{array}{l} n = 2^k \\ \log n = \log 2^k \\ k = \log_2 n \end{array} \right.$$

$$T(n) = O(\underline{n . \log \log n})$$

Q $T(n) = 4T(n/4) + cn^2$ — ①

Sol:- Backward substitution

In this problem, we replace n with $n/4$, $n/16$ respectively in various steps, to obtain a general expression

So at first step,

from ①

replace n with $n/4$

$$T(n/4) = 4T\left(\frac{n/4}{4}\right) + c(n/4)^2$$

$$\Rightarrow 4T(n/16) + cn^2/16$$

Sub this in ①

$$T(n) = 4T(n/16) + cn^2/16 \quad — ②$$

Next step is,

replace with $n/16$ in ①

$$T(n/16) \Rightarrow 4T\left(\frac{n/16}{4}\right) + c(n/16)^2$$

$$\Rightarrow 4T(n/64) + cn^2/256$$

Sub this in ②

$$T(n) = 16\left[4T(n/64) + cn^2/256\right] + \frac{4 \times cn^2}{16} + cn^2$$

$$\Rightarrow 64T(n/64) + \frac{16 \times cn^2}{256} + \frac{4 \times cn^2}{16} + cn^2 \quad — ③$$

here we divide our problem into 4 sub problem of same sizes, so assuming that $n = 4^k$, for simplication we apply logarithm on both sides

$\therefore \log_2 n = \log_2 4^k$

$\therefore \boxed{k = \log_4 n}$

As mentioned earlier, the general expression obtained from ①, ② & ③

$$T(n) = 4^k \cdot T(n/4^k) + cn^2 \underbrace{\left(\frac{1}{4^{k-1}} + \frac{1}{4^{k-2}} + \frac{1}{4^{k-3}}\right)}_{\text{So for } k \text{ times}} \text{—④}$$

$\sum_{i}^{k-1} \frac{1}{4^i} \Rightarrow$ applying summation

Sum of terms (G.P)

$$\Rightarrow \left[\frac{1-(1/4)^k}{1-1/4}\right] \qquad \left[\frac{a(1-r^n)}{1-r}\right.$$

$r<1, a = r = 1/4$

$$\Rightarrow \frac{1 - 1/4^{\log_4 n}}{1-1/4} \qquad \left[4^{\log_4 n} = n\right.$$

$$\Rightarrow \frac{\frac{n-1}{n}}{3/4} \Rightarrow \frac{4}{3}\left[1 - 1/n\right]$$

Sub this in ④

$$\therefore 4^k \cdot T(n/4^k) + cn^2 \times \frac{4}{3}\left[1 - 1/n\right] \text{—⑤}$$

So taking base

$T(n/4^k = 1) \Rightarrow 0$ [best case

$\therefore T(n = 4^k)$, $\boxed{k = \log_4 n}$

Sub this in ⑤

$$T(n) = n \times T(n/n) + cn^2 \times \frac{4}{3}\left[1 - 1/n\right]$$

$$T(n) = n \times T(1) + \frac{4}{3} cn^2 \left[1 - \frac{1}{n}\right]$$

$$= n \times 1 + \frac{4}{3} cn^2 \left[1 - \frac{1}{n}\right] \qquad [T(1) = 1]$$

$$T(n) = \Theta(n^2)$$

References :

1. E.Horowitz, S.Sahni, S.Rajasekaran, Fundamentals of Computer Algorithms, Galgotia Publications.

2. T.H. Cormen, C.E. Leiserson, R.L.Rivest, C.Stein, Introduction to Algorithms, PHI.

3. Sara Baase, A.V.Gelder, Computer Algorithms, Pearson.

4. Mark Allen Weiss, Data Structures and Algorithms in C , Pearson Publications.

5. Alfred V. Aho, Jon E. Hopcroft, Jeffrey D. Ullman, Design & Analysis of Computer Algorithms, PEarson Publications.

6. www.cse.iiitdm.ac.in/lectnotes.html